# HOMEWORK 6
# 415G 001 COMBINATORICS AND GRAPH THEORY

DUE FRIDAY 10/21

## Exercises

**1.** A *plane tree* is a rooted tree (it has one special node called the *root*) defined recursively as follows:
- A single vertex $\bullet$ (the root) is a plane tree.
- If we attach a new root connecting to all the roots of an ordered sequence $(P_1, P_2, \cdots, P_k)$ of plane trees we obtain a plane tree.

Figure 1 shows all plane trees with 4 vertices. Prove that the number $P_n$ of plane trees with $n$ vertices is $C_{n-1}$ the $(n-1)$-th Catalan number by exhibiting a bijection with a family of known Catalan objects. (Hint: Find a bijection with Dyck paths or with Ballot sequences).

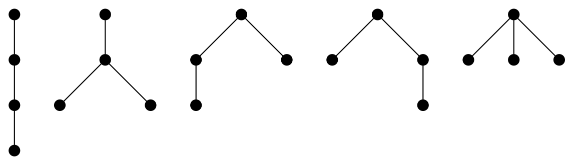
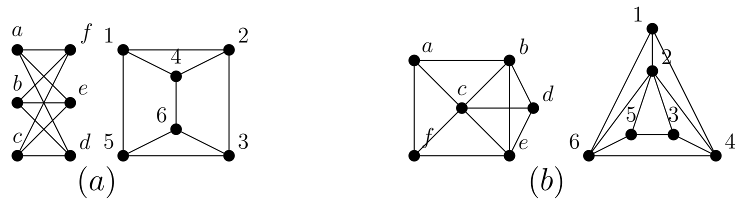


Figure 1. Plane trees with 4 vertices

**2.** Are the following pairs of graphs isomorphic? Explain why.




**3.** For a graph $G$ let $\overline{G}$ be the graph with the same vertex set as $G$ and with edge set satisfying $\{x, y\} \in E(\overline{G})$ if and only if $\{x, y\} \notin E(G)$. $\overline{G}$ is called the *complement* of $G$.

(a) If a graph $G$ has $n$ vertices, all of which but one have odd degree, how many vertices of odd degree are there in $\overline{G}$?

(b) If $G$ is an $n$-vertex graph that is isomorphic to its complement $\overline{G}$. How many edges does $G$ have?

**4.** Suppose $x$ and $y$ are the only vertices of odd degree in a graph $G$, and $x$ and $y$ are not adjacent to each other. Show that $G$ is connected if and only if the graph obtained from $G$ by adding the edge $\{x, y\}$ is connected.

## Suggested exercises

**Additional.**

**1.** A graph is called *regular* if all vertices have the same degree. Find all nonisomorphic regular simple graphs with four and five vertices.

**2.** Suppose all vertices of a graph $G$ have degree $d$, where $d$ is an odd number. Show that the number of edges of $G$ is a multiple of $d$.

**From the book.** 3.1, 3.2, 3.3, 3.4, 3.5, 3.6