Exercise Session 1

Combinatorial information in Flow Polytopes

Sage Days 130.5 – May 4 – May 8, 2026, Montréal

Permutation Flows

This session pairs hand computations on small flow polytopes with similar computations performed in SageMath. The goal is to build intuition for the combinatorial information encoded in flow polytopes, and to get comfortable with the tools for working with them. Coding problems live in the companion notebook `coding_1.ipynb`.

Manual problems

Problem 1. Find a formula for the number of vertices of $\mathcal{F}_G$ where G is the complete graph K_{n+1} on $[0, n]$ and $\mathbf{a} = \mathbf{e}_0 - \mathbf{e}_n$.

Problem 2. Find a formula for the number of vertices of $\mathcal{F}_G$ where G is the zig-zag graph $Zig(n)$ on $[0, n]$ with edge set

$$E(Zig(n)) = \{(i, j) \mid i < j \text{ and } j - i \leq 2\}.$$

[Hint: Can you find a recurrence?]

Problem 3. For the oruga graph $G = oru(2, 1)$ on three vertices, with three edges of the form $(0, 1)$ and two edges of the form $(1, 2)$, draw a sketch of a 3-dimensional projection of $\mathcal{F}_G(\mathbf{e}_0 - \mathbf{e}_2)$ and list its vertices.

Problem 4. Recall that the Pitman-Stanley graph PS_n is the graph on $[0, n]$ with edge set

$$E(PS_n) = \{(i, j) \mid i < j \text{ and } j = i + 1 \text{ or } j = n\}.$$

Use Hille's theorem to describe the face poset of $\mathcal{F}_{PS_n}(\mathbf{a})$ with $\mathbf{a} = \sum_{i=0}^{n-1} \mathbf{e}_i - n\mathbf{e}_n$. Conclude that $\mathcal{F}_{PS_n}(\mathbf{a})$ is combinatorially equivalent to $\mathcal{F}_{oru(n)}(\mathbf{e}_0 - \mathbf{e}_n)$ (the $(n-1)$-dimensional cube). [Hint: Two polytopes are combinatorially equivalent if they have isomorphic face posets.]

Problem 5. For CRY_5 with $\mathbf{a} = \mathbf{e}_0 - \mathbf{e}_n$, enumerate the integer flows indicated by the indegree Lidskii formula and confirm Zeilberger's formula (Chan-Robbins-Yuen conjecture) for the volume.

Problem 6. Use the indegree Lidskii formula to prove the formula for the volume of $\mathcal{F}_{oru(\mathbf{s})}(\mathbf{e}_0 - \mathbf{e}_n)$.

Problem 7. Draw all the 6 maximal cliques in $oru(3)$ for the two framings given in the lecture (the non-twisted and the twisted). Using these maximal cliques, draw the dual graphs of both triangulations.

Coding problems

The companion notebook `coding_1.ipynb` walks through the same constructions in SageMath. Open it with

```
sage -n jupyter
```

and use the answers from above to check your code.