Exercise Session 2

Permutation Flows and the Weak Order

Sage Days 130.5 – May 4 – May 8, 2026, Montréal

Permutation Flows Minicourse

This session moves from flow polytopes to the combinatorics of the DKK triangulation. We will work through the definitions and properties of the saturated cliques under the framing order, the lattice conjecture, the encoding by permutation flows, the weak order, and the G-Eulerian polynomial. Coding problems live in `coding_2.ipynb`.

Manual problems

Problem 1. Consider the following two framings of $G = \mathrm{car}(3) = \mathrm{car}(1, 1, 1)$.

car(3)

For both framings:

(a) Enumerate the saturated cliques $\mathcal{C}_1, \ldots, \mathcal{C}_k$ of $\mathrm{DKK}(G, F)$. [Hint: You should have $k = 5$ for each framing.]

(b) For every pair $\mathcal{C} \lessdot \mathcal{C}'$ sharing a facet, identify the minimal conflict — the descending route $P \in \mathcal{C}$ and the ascending route $Q \in \mathcal{C}'$.

(c) Draw the Hasse diagram of the framing order $P(G, F)$ and verify it is a lattice.

(d) Enumerate the set of saturated permutation flows $\mathrm{SatPermuFlows}(G, F)$.

(e) For each $\pi \in \mathrm{SatPermuFlows}(G, F)$, compute the final summary word at the sink.

(f) Compute the integer flow given by $(|\pi(e)| - 1)_{e \in E}$ and verify that these are precisely the lattice points of $\mathcal{F}_G(\vec{d})$.

(g) Draw the Hasse diagram of the weak order $\mathcal{W}(G, F)$ on $\mathrm{SatPermuFlows}(G, F)$. Verify that it is isomorphic to the framing order $P(G, F)$ on saturated cliques.

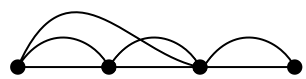
(h) Can you find a bijection between $\mathrm{SatPermuFlows}(G, F)$ for any of the framings and a set of σ-avoiding permutations for some pattern σ? What is that pattern? Can you find a bijection with Dyck paths or some other Catalan objects?

(i) Interpret the weak orders $\mathcal{W}(G, F)$ in terms of the combinatorics of these objects. For example, can you interpret the ascents and descents of $\pi \in \text{SatPermuFlows}(G, F)$ in terms of these objects?

(j) Calculate the (G, F)-Eulerian polynomials $A_{(G,F)}(t) = \sum_{\pi \in \text{SatPermuFlows}(G,F)} t^{\text{des}(\pi)}$.Verify that $A_{(G,F_1)}(t) = A_{(G,F_2)}(t)$ (independence of framing). Is this a familiar polynomial?

Problem 2. Compute the set of saturated permutation flows $\text{SatPermuFlows}(G, F)$ for $G = \text{oru}(1, 2, 1)$ where the framing F is given below.

oru(1, 2, 1)



Then calculate the following:

(a) Find a bijection between $\text{SatPermuFlows}(G, F)$ and the set of multiset permutations of the set $\{1, 2, 2, 3\}$ such that 1 cannot be located between the two 2s.

(b) Draw the Hasse diagram of the weak order $\mathcal{W}(G, F)$ on $\text{SatPermuFlows}(G, F)$.

(c) Can you interpret the ascents and descents of $\pi \in \text{SatPermuFlows}(G, F)$ in terms of these multiset permutations?

(d) Calculate the (G, F)-Eulerian polynomial $A_{(G,F)}(t) = \sum_{\pi \in \text{SatPermuFlows}(G,F)} t^{\text{des}(\pi)}$.

(e) Compute the Eulerian polynomial on the multiset permutations of $\{1, 2, 2, 3\}$ with the classical strict descent statistic. Does it match $A_{(G,F)}(t)$?

Coding problems

See `coding_2.ipynb`. You will use functions already coded in SageMath.